

Math 4200

Wednesday October 7.

2.4: We have a bit more to discuss in Monday's notes, and a fun quick theorem in today's. After that we'll discuss our first midterm on Friday.

Announcements:

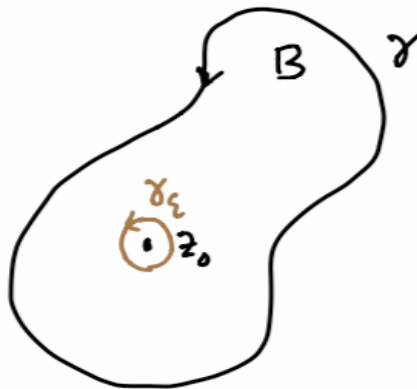
Math 4200-001
Week 7-8 concepts and homework
2.4
Due Friday October 16 at 11:59 p.m.

2.4 2, 3, 5, 7, 8, 12, 16, 17, 18. Hint: In problems 2, 5, 18 identify the contour integrals as expressing a certain function or one of its derivatives, at a point inside γ , via the Cauchy integral formulas for analytic functions and their derivatives.

w7.1 Prove the special case of the Cauchy integral formula that we discuss on Monday/Wednesday.

If γ is a counter-clockwise simple closed curve bounding a subdomain B in A , with z_0 inside γ , then the important special case of the Cauchy integral formula can be proven with contour replacement and a limiting argument, assuming f is C^1 in addition to being analytic:

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz.$$



First application of C.I.F. : f analytic implies f is infinitely differentiable, with estimates for the moduli of the derivatives. Rewrite the CIF as

$$f(z)I(\gamma; z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

(we replaced z_0 by z , and the contour integral variable z by ζ .)

Theorem 1 Let f be analytic in the open set $A \subseteq \mathbb{C}$, γ a p.w. C^1 contour homotopic to a point in A . Then for z inside γ , every derivative of f exists and may be computed by the contour integral formulas

$$f'(z)I(\gamma; z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

$$f^{(n)}(z)I(\gamma; z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

notice, these are the formulas we get by induction and "differentiating thru the integral sign" :

$$\frac{d}{dz} \frac{f(\zeta)}{\zeta - z} = f(\zeta) (-1) (\zeta - z)^{-2} (-1) = \frac{f(\zeta)}{(\zeta - z)^2}$$

$$\frac{d}{dz} \frac{f(\zeta)}{(\zeta - z)^n} = f(\zeta) (-n) (\zeta - z)^{-n-1} (-1) = n \frac{f(\zeta)}{(\zeta - z)^{n+1}}.$$

So, when can you justify this operation of differentiating thru the integral sign? That's an analysis question for after the midterm!

Corollary (*Liouville's Theorem*) Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be entire. Suppose f is also *bounded*, i.e. $\exists M \in \mathbb{R}$ such that $|f(z)| \leq M \quad \forall z \in \mathbb{C}$. Then f is constant.
proof: (It's very very short.)

Exam Friday October 9

email me (korevaar "at" math.utah.edu) your preferred two hour time slot on Friday, starting on the hour between 10:00 a.m. and 4:00 p.m.

I will email you a .pdf of the exam at the start time or a few minutes early. Unless you tell me otherwise I'll email it to the return address of the email you send me.

Complete the exam and upload a .pdf of your solutions to Gradescope by two hours after your start time. For insurance or if you have trouble uploading, email me a .pdf within your time limit as well.

The exam is closed book, closed notes, closed internet etc. Your only resource is yourself. I'll ask you to sign an honor-code like statement on the front of your exam which will be part of what you upload to Gradescope.

Potential Topics (we'll discuss):

Complex differentiability (def at a point, equivalent approximation formula, and "analytic" on a domain).

Cauchy Riemann equations

relationship to real differentiability, i.e. $f: \mathbb{C} \rightarrow \mathbb{C}$ analytic at z_0 is equivalent to what for $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ at (x_0, y_0) ?

consequences of definition of derivative and equivalent approximation formula

sum, product, quotient rules

chain rule

chain rule for curves

differential map df_{z_0}

using chain rule for curves to write CR in different coordinate systems.

inverse function theorem

harmonic functions and harmonic conjugates in simply-connected domains

Complex transformations

polar form for complex multiplication, powers, exponentials, logarithms.

$f(z) = az + b$, z^n , e^z , $\log z$, z^a , $\cos z$, $\sin z$, compositions

branch points, branch cuts, branch domains for root functions, logarithms, and compositions

Contour integration

definition and computation

relation to real-variables line integrals

Green's Theorem for contour integrals around domains (including domains with holes).

contour replacement for C^1 analytic integrands $f(z)$, via Green's Theorem and CR equations (Section 2.2 Cauchy's Theorem and Deformation/Replacement Theorem.)

estimates for modulus of contour integrals.

FTC

evaluation of contour integrals when the integrand is analytic, using FTC and/or contour replacement.

Homotopy-related ideas

homotopies

fixed endpoint

of closed paths

simply-connected domains

Antiderivatives of analytic functions

equivalence to path independence

local anti-derivative theorem, using rectangle lemma

global antiderivatives in (open) simply-connected domains, using homotopy lemma to prove path independence

Deformation Theorems via the homotopy lemma

for contours with fixed endpoints

for closed curves (section 2.3 Cauchy's Theorem)